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Magneto-charge-transport from twisting arguments

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Abstract. We use a model of constant negative curvature Riemann surfaces to study the effect of coupling together several Hall systems on the Hall conductances of the individual systems. We find that on average, the coupling does not effect the conductances. The method we use is based on the fact that the average conductances are given by Chern numbers, and hence are stable when one varies parameters of the Hamiltonian, as long as eigenvalues do not cross. The parameters we change are the surface parameters, and we find the averaged conductances of the surface by 'twisting' and 'pinching' it. This method is applicable to other systems as well.

1. Introduction

We study *coupled* Hall systems. Specifically, we want to know whether connecting together several Hall systems, or connecting Hall systems to a reservoir, effects the Hall conductance of each 'individual system'.

We shall not try to solve this problem in a fully general way, but only within the framework of a toy model which captures some of the properties of the general setup, and has the advantage of being (almost) solvable. Also we believe that our approach can be applied to other physical problems, being very general and physically intuitive.

The paper is organized as follows. In section 2, we give a very brief review of the integer quantum Hall effect, and present the toy model we work with. In section 3, we recall some known results which we use in section 4 to calculate the averaged Hall conductance. In section 5 we remark on generalizations of the results to other settings, and section 6 contains a summary and discussion.

Throughout this paper we adopt the system of units $\hbar = c = 2m_e = 1$, and absorb the electron's charge in the definition of the magnetic field B, and the fluxes ϕ_i. In these units, ϕ_0 (one flux quantum) is 2π.

2. Hall experiments, topology and Riemann surfaces

We review some known models for quantum Hall systems, add some new remarks, and explain our motivation in choosing our specific model for studying coupled Hall systems.

2.1. The quantum Hall experiment

Schematically, a laboratory Hall experiment is described as follows: take a sample, impose a voltage drop, and measure the current in a direction *perpendicular* to the voltage drop applied. (See figure 1(a). (There are other possible combinations of 'imposed' and

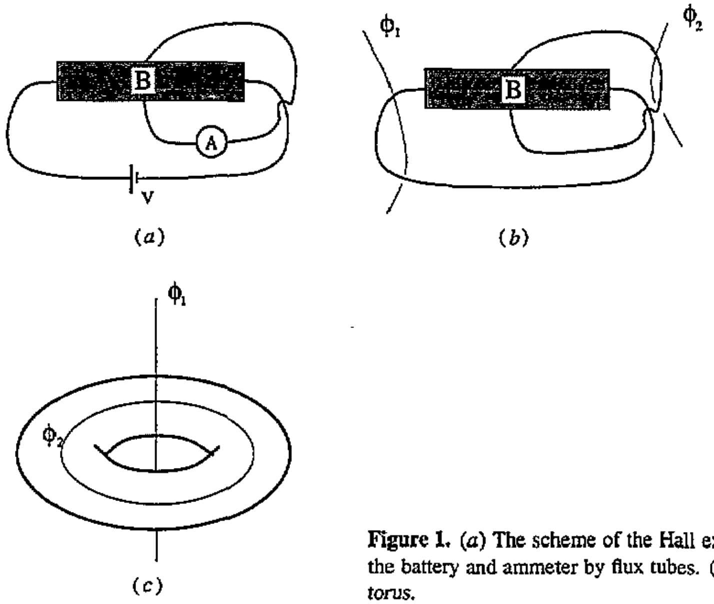

Figure 1. (*a*) The scheme of the Hall experiments. (*b*) Replacing the battery and ammeter by flux tubes. (*c*) Hall experiment on the torus.

'measured' quantities. This is irrelevant for us here). In the *integer* quantum Hall effect scenario, the conductance, I/V, is an integer multiple of $e^2/2\pi$. In this paper, we only discuss this case.

The integer quantum Hall effect (IQHE) was discovered in 1980 by von Klitzing *et al*, at low temperatures in samples which contained impurities [1].

A first theoretical model for explaining the quantization of the Hall conductance was given by Laughlin [2]. He considered a cylindrical sample, and gave an argument for the quantization of the *averaged* Hall conductance, where the average is taken over the values of an 'auxiliary' flux tube threading the cylinder (observe that, up to a unitary transformation, the Hamiltonian of a system is periodic in flux parameters). The argument is based on a 'single electron theory', and neglects interactions among electrons. This fact characterizes all the theoretical models of IQHE. Laughlin's argument suggests an integer conductance for every set of states belonging to a part of the spectrum which is separated by a gap from the rest of the spectrum of *non-localized* states, for all values of the flux parameter. The disorder potential causes the localization of most of the states, and hence is important for explaining the 'plateaux' found in the experiment.

An argument for the independence of the Hall conductance of the specific geometry of the sample was given by Thouless and co-workers (see, for example, [3]). The argument starts with the Kubo formula for the conductivity, and is based mainly on the exponential decay of the Green functions in a disordered system.

Thouless *et al* (TKNN), [4] considered the case of an infinite sample with a periodic potential. For rational values of the magnetic flux through a unit cell, $\phi = \phi_0\, p/q$, each Landau level splits into q sub-bands. Using the Kubo formula, TKNN proved that the Hall conductivity of each sub-band is an integer. Moreover, they proved (without stating it in so many words), that this integer is a *topological invariant*, being a Chern number of a $U(1)$ line bundle over the k torus, (k stands for the Bloch momentum vector). One gets this integer after integration on all k's in the Brillouin zone. In other words only a *full* band carries an integer conductance. This is a trivial but important observation for what follows. Also note that TKNN proved the quantization of the Hall conductance, and not of its average, as in Laughlin's argument.

A different approach was introduced by Avron and Seiler: they pointed out that if the leads are considered as part of the quantum system† and one replaces the voltage source by an adiabatically time varying flux tube (say ϕ_1) and put an auxiliary flux (ϕ_2) threading the ammeter 'loop', then the charge transport through the ammeter loop during an increase of ϕ_1 by one flux quantum, *averaged* over all possible values of ϕ_2, is given by a Chern number (which is, by definition, an integer). The disadvantage of this approach is that *one can prove quantization only for 'double averaged' quantities.* (Although one may use Thouless' arguments to claim that this is of no importance if there is disorder in the system). The advantage is that this model can be generalized to many different settings, for example, the model we shall discuss in this paper.

Now, instead of the experimental setting shown in figure 1(a) take the 'gedanken' experiment setting given in figure 1(b): a torus threaded by two flux tubes. For this setting, the same calculation used by Avron and Seiler to prove the quantization of the averaged Hall conductance goes through, and we get the quantization on the torus with arbitrary background potential. Let us also assume that the torus is *flat*. Then, it is easy to understand both the quantization and the need for averaging. Each value of the fluxes threading the torus represents a certain value of k if one 'unfolds' the torus to get its covering space, the Euclidian plane with a periodic potential (now with an integer magnetic flux through each unit cell). This is exactly the TKNN setting. The average is not needed in the special case that the value of the conductance is independent of the fluxes, or, in other words, if the bands in the periodic potential case are flat (k independent). This is so in the case of a vanishing potential, and indeed, the Hall conductance of a full Landau level on the flat torus is *one*, without averaging.

2.2. *Selecting a model*

Can we take the model of electrons on a flat torus as a 'good' model for the IQHE? On the one hand, from Thouless' arguments, if one is interested in the conductance as defined by Kubo's formula, and the sample is large enough, the conductance of the bulk should not depend on the shape of the sample, so we might as well take it to be a cylinder (as Laughlin did), or a torus. On the other hand, a different model is better—that of a *punctured* torus. Assuming we consider the leads as a part of the quantum system. It is then natural to take them to be two-dimensional, such that the connection between 'a lead' and 'the sample' is smooth. The corresponding Hall system is drawn in figure 2(a) (one should identify opposite edges). Notice that naturally, we apply a magnetic field on the leads too. It is easy to see in figure 2(a), and even easier in the equivalent to it, figure 2(b), that the topology of the sample is that of a 'punctured torus'—a torus with a hole.

Moreover, on a punctured torus, unlike the torus, one can apply a magnetic field which takes an arbitrary value (due to the fact that a Dirac string may enter through the puncture). Also, there is no need for a magnetic monopole for applying a constant magnetic field on the surface. Hence, this model is closer to reality.

The 'theoretical disadvantage' of this model comes from the boundary: it seems impossible to study analytically the Landau Hamiltonian on the punctured torus of figure 2(a), if one wants to impose Dirichlet (or Neumann) boundary conditions. To overcome this we pick a punctured torus without a boundary: a *constant negative curvature* punctured torus. The boundary conditions are replaced by L^2 (square integrability) conditions.

† Notice that the system is now *compact*, and the number of electrons in it is constant.

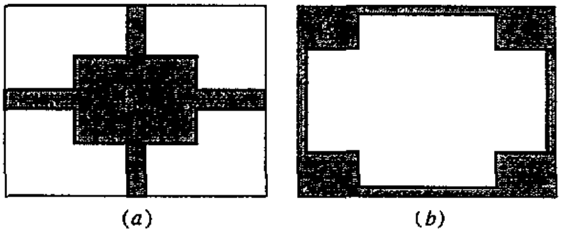

Figure 2. (*a*) A punctured torus (opposite sides are identified). (*b*) The same punctured torus, drawn differently.

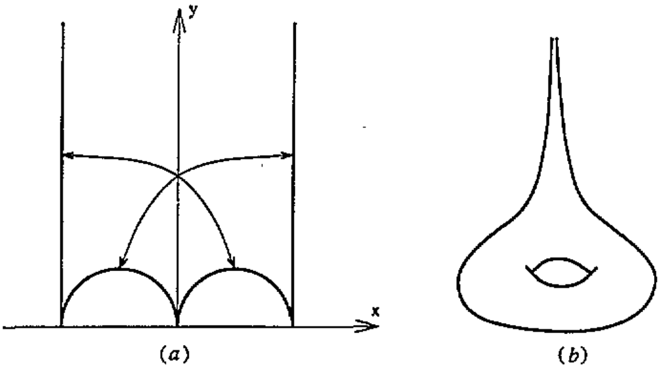

Figure 3. (*a*) A fundamental domain of a constant negative curvature punctured torus. (*b*) The topology of the surface obtained after the identifications.

As the flat torus is described as R^2/Z^2, the punctured torus is H/Γ, where H is the Poincaré upper half plane, and Γ is a discrete subgroup of $SL(2, R)$, the group of transformations which leave H invariant. A *fundamental domain* for the punctured torus is given, for example, in figure 3(*a*). Notice that since the metric on the upper half plane is given by $\mathrm{d}s^2 = ay^2(\mathrm{d}x^2 + \mathrm{d}y^2)$, the boundaries of the fundamental domain are geodesics, and all of them are of infinite length. Because a appears only in an overall factor, a^{-2}, of the energy eigenvalues, in the following we consider the case $a = 1$. Modification to any other value is trivial.

A schematic drawing of the surface obtained after the identifications is given in figure 3(*b*). Notice that although the surface is non-compact, its area is finite. This goes well with the fact that the Hall system is not compact (being connected to a current source or to a battery), but the sample itself has a finite area.

A 'Hall experiment' in this model would involve changing one of the two fluxes which thread the handle, and measuring the current 'around' the second.

Now, that we have both the flat torus and the punctured torus models for the Hall experiment, we would like to have a model for *coupled* Hall systems, for example, if we want to couple two punctured tori, we can do it in two ways: we can glue together the two punctures, and obtain a compact surface of genus-2, or we can cut a circle from each, and connect them by a cylinder, getting a genus-2 surface with two punctures. We can also model connections to a reservoir by adding punctures. By the uniformization theorem [5], all the surfaces which satisfy $2g + h \geqslant 3$, can be described as H/Γ for some Γ.

Hence, the model we choose for coupled Hall systems is that of a Schrödinger particle in a constant magnetic field on a genus g, finite area Riemann surface (g is the number of 'handles'—for the sphere—$g = 0$, for the torus—$g = 1$ etc). The surface may either be compact, or have h punctures at infinity (and h infinitely long 'horns' leading to them).

We consider the free case only, and calculate the average Hall conductance of a full 'Landau level'. To prove existence of plateaux one should also prove the localization of most of the states in the presence of a random potential.

3. Some known results

The model we consider, a Schrödinger particle in a constant magnetic field on a genus g, finite area Riemann surface, was studied initially in [6]. This model has the advantage that the whole system, including the leads, is described by the Schrödinger operator. One can thread the system by $2g+h$ Aharonov–Bohm fluxes—$2g$ through the handles, and h through the horns. The adiabatic (in this case, non-averaged) charge transport due to a variation of the h *horn fluxes* was calculated in [6]. This calculation teaches us nothing about the question of coupled Hall systems. Indeed the horn fluxes do not have an 'experimental set-up' analogue, as one cannot put flux tubes threading the pseudo-one-dimensional leads. Here we want to study the *averaged* charge transport due to the variation of the $2g$ handle fluxes. We note that our results, combined with the results of [6], give complete knowledge on *averaged* adiabatic transport properties of finite area surfaces of constant negative curvature.

In this section, we summarize the results of [7, 6] we need.

Given a constant negative curvature (-1), smooth Riemann surface of genus g with h cusps, we can thread it with Aharonov–Bohm fluxes of three types: $2g$ fluxes through the handles, denoted by ϕ_j^g, $j = 1, \ldots, 2g$, h fluxes which thread the horns, denoted by ϕ_j^h, $j = 1, \ldots, h$, and an arbitrary number of fluxes piercing the surface. In the following, we assume that the piercing fluxes vanish. On the surface we apply a constant magnetic field $B > 0$ (the assumption of positivity is only for convenience: all the analysis carries through for negative B as well).

As was shown in [7], the charge transport around a flux ϕ_k, (averaged over all values $0 \leqslant \Phi_k < 2\pi$), during the adiabatic increasing of a different flux ϕ_l by one flux quantum, is given by a Chern number: $c_{kl} = -(\mathrm{i}/2\pi)\iint \mathrm{Tr} P\, \mathbf{d}P \wedge \mathbf{d}P$, where P is a spectral projection on part of the spectrum, which is separated by gaps from the remainder of the spectrum for any values of ϕ_k and ϕ_l, $\mathbf{d}P = \sum_i (\partial P/\partial\phi_i)\mathrm{d}\phi_i$, and the integration is over the torus of fluxes: $0 \leqslant \phi_k,\ \phi_l < 2\pi$. For the experimental set-up of the QHE, this averaged charge transport is proportional to the averaged Hall conductance. In our units, the constant of proportionality is 2π [7].

The c_{kl} are integers. Being topological invariants, they are stable under changes of parameters in the Hamiltonian, as long as energy levels do not cross. We shall make use of this stability in the following.

We need the following facts (see [6] and references therein):

(i) The magnetic field on the Riemann surface satisfies the Dirac quantization condition: $\int B - \sum_{j=1}^{h} \phi_j^h = 0 \pmod{2\pi}$.

(ii) The spectrum of the 'Landau levels' is

$$E_n(B, \phi_j^g, \phi_j^h) = B(2n+1) - n(n+1), \quad n = 0, 1, \ldots, \left[B - \tfrac{3}{2}\right]$$

where $x \leqslant [x] < x+1$ denotes the integer part of x. (Above the 'Landau levels', we do not know the spectrum in general. All we know is that up to a shift of B^2, it coincides with the spectrum of the 'free' Hamiltonian, with no applied magnetic field).

(iii) The degeneracies of the Landau levels are

$$D(E_n) = (2B - 2n - 1)\frac{A}{4\pi} + \frac{h}{2} - \sum_{j=1}^{h}\{\varphi_j^h\} - h^*$$
$$= \frac{1}{2\pi}\int B - \sum_{j=1}^{h}\{\varphi_j^h\} - (2n+1)(g-1) - nh - h^*$$

where $A = 2\pi(2g - 2 + h)$ is the area of the surface, $\varphi_j \equiv \phi_j/2\pi$ is the incoming flux through horn j, $0 \leqslant \{x\} < 1$ denotes the fractional part of x, and h^* is the number of cusps through which the incoming flux vanishes (such cusps admit scattering states. See [6, 8, 9]). By the Dirac quantization—$D(E_n)$ is an integer.

From these facts we learn that both the energies of the Landau levels, and their degeneracies, *do not depend on the fluxes threading the handles*, neither explicitly, nor implicitly (via the Dirac quantization condition). Moreover, the energy levels and the degeneracies do not depend on the specific surface we deal with, but only on its 'topology'. In other words, they are invariant within the Teichmüller space (which is, very roughly speaking, the space of all constant curvature surfaces having the same topology. For a more precise definition see, for example, [10]). This turns out to be (almost) all the needed information for calculating the Chern numbers associated with the handle fluxes: the energy levels and their degeneracies are constant within the Teichmüller space. This means that if we regard the 'coordinates' of the point in the Teichmüller space as additional parameters of the Hamiltonian—their change does not effect c_{ij}.

4. The average Hall conductances

In order to use the invariance for calculating conductances let us introduce a convenient parametrization of the Teichmüller space, known as the Fenchel–Nielsen coordinates. More details can be found in [10, 11]. This parametrization enters naturally if one builds a Riemann surface by pasting together 'pairs of pants'. 'A pair of pants' is, topologically, a sphere with three holes. It has a constant negative curvature -1, and its boundaries are geodesics. We denote the lengths of the boundaries by l_i. If $l_i = 0$, we get a cusp. It is known that a pair of pants exists for any choice of lengths (l_1, l_2, l_3). Any smooth, constant negative curvature Riemann surface can be built by pasting together $2g+h-2$ pairs of pants.

For example, we demonstrate how to build a $g = 2$, $h = 1$ Riemann surface: we take $2g + h - 2 = 3$ pairs of pants, with lengths of boundary geodesics: $L_1 = (0, l_1, l_2)$, $L_2 = (l_1, l_2, l_3)$, $L_3 = (l_3, l_4, l_4)$ (see figure 4). Then we glue together the 'matching' boundaries. But there is a freedom: one can rotate the boundaries relative to each other before the gluing. More formally: let us parametrize the boundary geodesics by an angle $t \in S^1$. Then, we can paste $\gamma(t) = \gamma'(\alpha - t)$, where γ and γ' are two boundary geodesics of equal length, which we paste; α is called 'a *twist* parameter'.

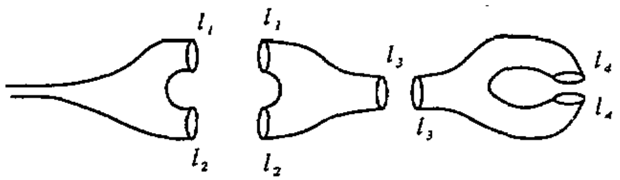

Figure 4. A $g = 2$, surface can be build from three pairs of pants.

In our example we have four twist parameters. The Fenchel–Nielsen coordinates for a given Riemann surface are the lengths of the boundary geodesics of the corresponding pants, and twist parameters. In general, the number of parameters (which is the dimension of the Teichmüller space), is $6g - 6 + 2h$. (In our example 8). Half of them are twist parameters. Notice that there are two 'types' of these pants boundary geodesics: geodesics such that cutting the surface along them splits the surface into two disconnected pieces (these are the $g-1$ geodesics around the 'neck'; in our example, the geodesic of length l_3), and there are geodesics which do not separate the surface into two parts (the $2g - 2 + h$ geodesics around the handles; in the example l_1, l_2 and l_4).

As a matter of convenience, we denote the two fluxes through the handle 'i' by ϕ_i, ϕ_{i+g}, $i = 1, \ldots, g$ (and call the corresponding loops 'loop i' and 'loop $i+g$', $\oint_j A = \phi_j$). Because we only deal with handle fluxes we omit the superscript g of ϕ.

Now we use the facts we already know to show that $\forall i, (1 \leqslant i \leqslant g)$, $|c_{ij}| = \delta_{j,i+g}$. (The Chern numbers associated with two fluxes threading the same handle are either $+1$ or -1, and all other Chern numbers vanish.)

- (i) We take a Riemann surface with g handles, which is threaded by two handle fluxes, ϕ_i, ϕ_j and $j \neq i+g$, and all the other handle fluxes vanish. (The fluxes through the cusps need not vanish). We cut the surface into two pieces along a geodesic which separates the two handles threaded by ϕ_i and ϕ_j. Now we can 'twist' the piece containing ϕ_j by any angle α we want, and then glue it again. This gives a family of surfaces, and a corresponding family of Hamiltonians, parametrized by the twist parameter α. The Chern numbers for the 'original' surface, $c_{ij}(0)$, are identical to the Chern numbers for the 'twisted' surfaces, $c_{ij}(\alpha)$ (because we know that during the twist there is no level crossing). Now assume that we start with a 'symmetric' surface, such that after a twist by π we get a surface which is *identical* to it. Then, because a twist of π 'sends' ϕ_j to $-\phi_j$, while keeping ϕ_i unchanged, $c_{ij}(\pi) = -c_{ij}(0)$. (From physical intuition the direction of the electromotive force due to the adiabatic change of ϕ_j is reversed, and hence the charged transport must reverse its orientation, too). From this, we conclude that $c_{ij}(\pi) = c_{ij}(0) = -c_{ij}(0)$, or $\forall\alpha$, $c_{ij}(\alpha) = 0$.
  Notice that although we only gave a proof for a 'symmetric' surface and vanishing handle fluxes (other than ϕ_i and ϕ_j) the result holds for any surface and any values of handle fluxes because the variation of the surface and the handle fluxes does not change c_{ij} (there is no level crossing during the variation). More details can be found in the appendix.
- (ii) We concentrate on one handle and study the charge transport around loop $i+g$ during an adiabatic increasing of ϕ_i by one flux quantum.
  To do this we treat l_i (the length of loop i) as a parameter, and shrink it. In the limit, as $l_i \to 0$, we get a surface of a different topology (genus $g-1$, and $h+2$ cusps; see figure 5), but, generically (for $\phi_i \neq 0$) with the *same* Landau level energy spectrum and degeneracies.
  As was analysed in [6], for this degenerate surface during the adiabatic increase of ϕ_i by one flux quantum, exactly one state per Landau level is transported from one cusp to the other, or, in other words encircles the loop $i+g$. Hence, for $j = i+g$, $|c_{ij}| = 1$. (The sign depends on the relative orientation of loops i and $i+g$. Without loss of generality, we take a relative orientation such that $c_{i,i+g} = +1$).

Notice that our results hold both for compact surfaces, and for non-compact ones ('leaky tori').

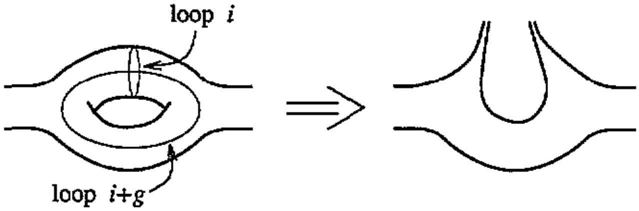

Figure 5. A degeneration of a non-seperating geodesic reduces the genus and creates two cusps.

5. Generalizations

An immediate generalization of the above is to admit e 'elliptic points' (conic points of integer order) on the surface. The energies of Landau levels remain the same. The degeneracies are modified to (see [8, 12]):

$$D(E_n) = \frac{1}{2\pi}\int B - \sum_{j=1}^{h}\{\varphi_j^h\} - \sum_{j=1}^{e}\left(\left\{\varphi_j^e + \frac{n}{\nu_j}\right\} - \frac{n}{\nu_j}\right) - (2n+1)(g-1) - n(h+e) - h^*$$

where ν_j denotes the order of the jth elliptic point, and φ_j^h, φ_j^e denote incoming fluxes through a cusp or an elliptic point, respectively. The modified Dirac quantization condition is: $(1/2\pi)\int B - \sum_{j=1}^{h}\{\varphi_j\} - \sum_{j=1}^{e}\{\varphi_j^e\} = 0 \pmod 1$, and hence $D(E_n)$ is indeed an integer. All the previous analysis can be carried out for this case, too, and we get the same Chern numbers.

Now we want to vary the *curvature* of the surface. If we do so, while keeping the magnetic field constant (i.e. proportional to the area form) the ground state has energy B, and its degeneracy, on a compact Riemann surface, is $D_0 = (1/2\pi)\int B + (1-g)$ as long as $B > |k|$ everywhere, where k denotes the Gaussian curvature of the surface (see, for example, [13]). Arguments similar to those we used again give the same Chern numbers.

We note that for this case (a smooth, compact, Riemann surface), one can also find an explicit expression for the non-averaged conductance. This was done in [14], and the result shows that the conductance, and also the charge transport, fluctuate as a function of the fluxes.

For more general surfaces, all we can say is that if $c_{ij}(P)$ are well defined, where P is the projection on the ground state then $c_{ij} = c\,\delta_{i+g,j}$, where c is a constant which is independent of i. (*Sufficient* conditions for this are that the surface has finite area, and the magnetic field is high 'enough', compared to the Gaussian curvature. We do not yet know what conditions are necessary). This is due to the fact that we can interchange two handles i and i' by a smooth deformation of the surface, without changing the ground- state energy or degeneracy, hence without changing c_{ij}. Hence, $\forall i, i'$, $c_{i,i+g} = c_{i',i'+g} \equiv c$ (as before, $c_{ij} = 0$ for $j \neq i+g$ follows trivially from our twisting argument).

As another application of 'twisting arguments' we can prove that the c_{ij} vanish on a symmetric *graph* with a *separation vertex* (removing such a vertex splits the graph into two components), whenever i and j are in different 'sides' of this vertex. This follows from the fact that twisting one component relative to the other around the separation vertex does not change the Hamiltonian. More details on quantum mechanics and Chern numbers on graphs can be found in [15].

6. Summary

We found that on average the charge transport on Riemann surfaces is *local*—the averaged transport around a handle vanished unless there was an electromotive force acting on it, in which case, the averaged transport is in the direction perpendicular to the EMF. In other words the averaged conductance resembles the expected 'classical' result, and does not depend on the global properties of the surface. However, due to [14], one knows that this is no longer correct for the non-averaged quantities. We note that only the average quantity is stable against perturbations. Moreover, Thouless' arguments suggest that in reality, when impurities are present, there will only be exponentially small fluctuations. Hence, we conclude that coupling together Hall systems, or coupling Hell systems to a reservoir, does not affect the conductance of each individual system.

We note to prove that Thouless' arguments [3] hold for a general surface is left as an open problem.

Appendix.

In the following, we elaborate more on two central points in our proof.

- On a 'symmetric' surface, 'twisting' a handle by π reverses the signs of the two fluxes threading it.
  To prove this, we introduce the *Schottky uniformization* of a Riemann surface [16]: a genus g Riemann surface can be described as the Riemann sphere ($C \bigcup \infty$), with $2g$ discs removed. The boundaries are pairwise identified. We denote the identifying transformations by σ_i. To introduce cusps one starts with a 'punctured' Riemann sphere. For example, see figure 6: the identifications σ_i, σ_j give a sphere with two handles—a $g = 2$ surface.

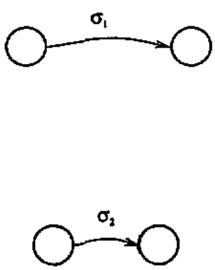

Figure A1. A Schottky uniformization for a $g = 2$ surface.

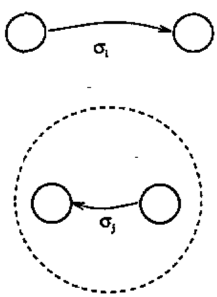

Figure A2. A Schottdy uniformization for the 'twisted' surface.

  We want to consider the effect of a twist on the fluxes. There are two types of twists. (i) A twist within a handle (around a 'non-separating' geodesic): such a twist is obtained by modifying the σ's, such that different points are identified. (ii) A twist of a handle relative to the others: to perform such a twist, we 'cut' a circle around two holes in the Riemann sphere (which, under the identification σ_j, form the handle), rotate this circle by the desired amount, and 'glue' back We are interested in the rotation by π. The 'new' uniformization, after the twist, is given in figure 7: we see that σ_j reversed its direction, and the circles it identifies reversed their orientation.

To find the effect of this twist on the fluxes we introduce them to the model: ϕ_i and ϕ_{i+j} are given by the line integrals $\oint \mathrm{A}$ along the homology cycles i (formed by the application of σ_i) and $i+g$ (along one of the circles which are identified by σ_i). From these definitions, and the stated above, it is clear that 'twisting' the ith handle by π flips the sign of ϕ_i, ϕ_{i+g}.

- We have two *identical* surfaces (the 'original' and the 'twisted' surfaces), which are each threaded by two flux tubes, such that $\phi_i' = \phi_i$, $\phi_j' = -\phi_j$ (we denote by a prime a parameter, on an operator, associated with the twisted surface). Hence, $P'(\phi_i', \phi_j') = P(\phi_i', \phi_j') = P(\phi_i, -\phi_j)$. We want to prove that $c_{ij}' = -c_{ij}$, where

$$c_{ij} = -\frac{\mathrm{i}}{2\pi} \int_{\phi_i=0}^{2\pi} \int_{\phi_j=0}^{2\pi} \mathrm{Tr}\, P\, \mathbf{d}P \wedge \mathbf{d}P \qquad c_{ij}' = -\frac{\mathrm{i}}{2\pi} \int_{\phi_i'=0}^{2\pi} \int_{\phi_j'=0}^{2\pi} \mathrm{Tr}\, P\, \mathbf{d}P \wedge \mathbf{d}P .$$

Using the fact that $\mathbf{d}P(\phi_i', \phi_j', \ldots) = \mathbf{d}P(\phi_i, \phi_j, \ldots)$ we obtain

$$c_{ij}' = -\frac{\mathrm{i}}{2\pi} \int_{\phi_i=0}^{2\pi} \int_{-\phi_j=0}^{2\pi} \mathrm{Tr}\, P\, \mathbf{d}P \wedge \mathbf{d}P = +\frac{\mathrm{i}}{2\pi} \int_{\phi_i=0}^{2\pi} \int_{\phi_j=-2\pi}^{0} \mathrm{Tr}\, P\, \mathbf{d}P \wedge \mathbf{d}P = -c_{ij} .$$

□

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